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Ballots and Trees

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ABSTRACT

The parking problem described in the first paragraph below attracts interest because of the unexpected appearance of the numbers of (free) trees with all points labeled. This paper reports first a simple mapping, due to H. O. Pollak, of the preference sets of the parking problem on the trees. Then a relation of the parking problem to ballot problems is examined.

1. Consider the following parking problem: n parking spaces are arranged in a line, numbered 1 to n left to right; n cars, arriving successively, have initial parking preferences, p_i for car i, chosen independently and at random; if space p_i is occupied, car i moves to the first unoccupied space (if any) to the right. In how many of the n^n preference sets are all cars parked? The answer turns out to be $(n+1)^{n-1}$, which by Cayley's formula is T_{n+1}, the number of (free) trees with $n+1$ labeled points. In this note I give a simple proof of this, due essentially to my friend and colleague H. O. Pollak, his mapping of the preference sets on trees, and a second proof whose chief attraction is an association with ballots.

2. The given preference set is $(p_1, ..., p_n)$, $p_i = 1(1)n$. Consider the related set $(p_1, ..., p_n)$, $p_i = 1(1)\overline{n+1}$, with spaces 1 to $n+1$ arranged clockwise on a circle. Preference $p_i = n+1$ is treated like any other preference: if space $n+1$ is occupied, car i moves clockwise to the first unoccupied space. Every set leaves one space unoccupied, and because of symmetry the number of sets leaving a given space, say k, unoccupied is the same for every k, $k = 1(1)\overline{n+1}$. Hence the number with $k = n+1$, which is the number with all cars parked, is $(n+1)^{-1}$ times the total number of sets, $(n+1)^n$, and so equals T_{n+1}. It is worth noting that with m cars parking, $m \leqslant n$, the same argument shows that the number of preference sets in which all cars are parked is $T(n+1, n+1-m)$ with

$T(n, k) = kn^{n-1-k}$. By a result due to Cayley [1] (brought to my notice by Moon [3]), $T(n, k)$ is the number of forests of labeled trees with n points and k trees, such that tree i contains point i.

3. Pollak's mapping is as follows. With each set $(p_1, \ldots, p_n)$, for which all cars are parked, associate the difference set $(\pi_1, \ldots, \pi_{n-1})$ with

$$\pi_i = p_{i+1} - p_i, \qquad \mathrm{mod}(n+1).$$

Then the set $(\pi_1, \ldots, \pi_{n-1})$ is a Prüfer coding of a tree with $n + 1$ points. A Prüfer coding [5] is the following: from the given tree, remove the smallest end-point and note the number a_1 of the (unique) point adjacent to this end-point. Repeat the procedure, obtaining numbers $a_2, a_3, \ldots$, until only two adjacent points remain. The set $(a_1, \ldots, a_{n-2})$ is the coding. As is well known, the coding is reversible.

4. For any preference set $(p_1, \ldots, p_n)$ the number parked is independent of the order of the numbers p_i. A character of the set independent of order is the preference partition: $1^{a_1}2^{a_2} \cdots n^{a_n}$, with a_i the number of cars preferring space i. It is clear that, if all cars are parked, $a_n \leqslant 1$, $a_{n-1} + a_n \leqslant 2$, $a_{n-j+1} + \cdots + a_n \leqslant j$, $j = 1(1)\overline{n-1}$, while $a_1 + a_2 + \cdots + a_n = n$. With $A_j = a_1 + \cdots + a_j$, these conditions are equivalent to $A_j \geqslant j$, $j = 1(1)\overline{n-1}$, $A_n = n$. Thus, for all cars parked, the number of preference partitions is the number of ways of putting n like balls into n different cells such that the sum of content of cells 1 to j is at least j.

To find this number it is convenient to consider the same problem with m different cells, $m \leqslant n$; that is, with $A_j \geqslant j$, $j = 1(1)\overline{m-1}$, $A_m = n$. If $N(n, m)$ is the number in question, then

$$N(n, m) = N(n, m-1) + N(n-1, m).$$

The proof is as follows. In the distributions in question, either cell m is empty, or it is not. If it is empty, $A_{m-1} = n$, which gives the first term on the right. If it is not, removal of one ball from cell m gives the conditions for $n - 1$ balls in m cells. As the recurrence is that of the classical ballot numbers

$$a_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1}$$

and the initial conditions are $N(n, 1) = 1 = a_{n0}$, $N(n, 2) = n = a_{n1}$, it is clear that

$$N(n, m) = a_{n,m-1},$$
$$N(n, n) = a_{n,n-1} = a_{nn} = (n+1)^{-1}\binom{2n}{n}.$$

The number a_{nm} is the number of paths on a (square) lattice from (0, 0) to (n, m) which do not cross the diagonal (weak lead election returns).

This recalls the result of Harris [2] (accomplished by an elegantly simple mapping) that a_{nn} is the number of rooted trees with $n+1$ points with branches at the root or at any other branch point counted as distinct if they are different trees.

Returning to preference partitions, it is clear that the set $(a_1, ..., a_n)$ is a composition of n with zero parts permitted. It may be mapped on a lattice path by the following rule: a_i is the (integral) horizontal segment length at level $i-1$; the paths go from (0, 0) to (n, n) without crossing the diagonal. The set of compositions with k positive parts corresponds to the set of lattice paths with $2k$ segments; by a result of Narayana [4] the latter is counted by

$$d(n, k) = n^{-1} \binom{n}{k}\binom{n}{k-1}.$$

A composition with the partition indicator $x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}$, that is, with k_i parts of size i, has a count proportional to $k!/k_1! \cdots k_n!$, with $k_1 + \cdots + k_n = k$. Since

$$\sum_{\pi(n)} \frac{k!}{k_1! \cdots k_n!} = \binom{n-1}{k-1}, \qquad k_1 + \cdots + k_n = k,$$

with $\pi(n)$ indicating summation over all partitions of n, the count for this composition is

$$d(n, k) \binom{n-1}{k-1}^{-1} \frac{k!}{k_1! \cdots k_n!} = \frac{(n)_{k-1}}{k_1! \cdots k_n!}.$$

Since a preference partition $1^{a_1} \cdots n^{a_n}$ corresponds to $n!/a_1! \cdots a_n!$ preference sets, it follows that the total number of preference sets such that all cars are parked is given by

$$\sum_{\pi(n)} \frac{(n)_{k-1}}{k_1! \cdots k_n! 1!^{k_1} \cdots n!^{k_n}} = Y_n(f, ..., f), \qquad f^k \equiv f_k(n) = (n)_{k-1}$$

with Y_n the Bell multivariable polynomial. Finally, with $S(n, k)$ a Stirling number of the second kind,

$$\begin{aligned} Y_n(f, ..., f) &= \sum_{k=0}^{n} S(n, k) f_k \\ &= (n+1)^{-1} \sum_{0}^{n} S(n, k)(n+1)_k = (n+1)^{n-1} = T_{n+1}. \end{aligned}$$

References

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